$x(t) = 5\cos(2\pi(3)t + 50^\circ) \Rightarrow f = 3$

$x(t) = 5\cos(at + b) \Rightarrow f = \frac{a}{2\pi}$

5.2 Instantaneous Frequency

Definition 5.9. The ***generalized sinusoidal*** signal is a signal of the form

$$x(t) = A\cos(\theta(t)) \tag{73}$$

Euler's formula

where $\theta(t)$ is called the ***generalized angle***.

- The generalized angle for conventional sinusoid is $\theta(t) = 2\pi f_c t + \phi$.
- In [3, p 208], $\theta(t)$ of the form $2\pi f_c t + \phi(t)$ is called the **total instantaneous angle**.

Definition 5.10. If $\theta(t)$ in (73) contains the message information $m(t)$, we have a process that may be termed **angle modulation**.

- The amplitude of an angle-modulated wave is constant.

$\cos(x) = \mathrm{Re}\{e^{jx}\}$

- Another name for this process is **exponential modulation**.
    - The motivation for this name is clear when we write $x(t)$ as $A\mathrm{Re}\left\{e^{j\theta(t)}\right\}$.
    - It also emphasizes the nonlinear relationship between $x(t)$ and $m(t)$.
- Since exponential modulation is a nonlinear process, the modulated wave $x(t)$ does not resemble the message waveform $m(t)$.

5.11. Suppose we want the frequency f_c of a carrier $A\cos(2\pi f_c t)$ to vary with time as in (72). It is tempting to consider the signal

It is tempting to think $g(t)$ plays the role of "freq." here.

$$A\cos(2\pi \underbrace{g(t)t}_{\theta(t)}), \tag{74}$$

$f(t) = \frac{1}{2\pi}\frac{d}{dt}\theta(t)$

$= \frac{d}{dt}\left((g(t))(t)\right) = g(t) + t\frac{d}{dt}g(t)$ extra term

$\neq g(t)$

where $g(t)$ is the desired frequency at time t.

Example 5.12. Consider the generalized sinusoid signal of the form 74 above with $g(t) = t^2$. We want to find its frequency at $t = 2$.

$A\cos(2\pi(t^2)t) = A\cos(2\pi t^3)$

(a) Suppose we guess that its frequency at time t should be $g(t)$. Then, at time $t = 2$, its frequency should be $t^2 = 4$. However, when compared with $\cos(2\pi(4)t)$ in Figure 38a, around $t = 2$, the "frequency" of $\cos(2\pi\left(t^2\right)t)$ is quite different from the 4-Hz cosine approximation. Therefore, 4 Hz is too low to be the frequency of $\cos(2\pi\left(t^2\right)t)$ around $t = 2$.

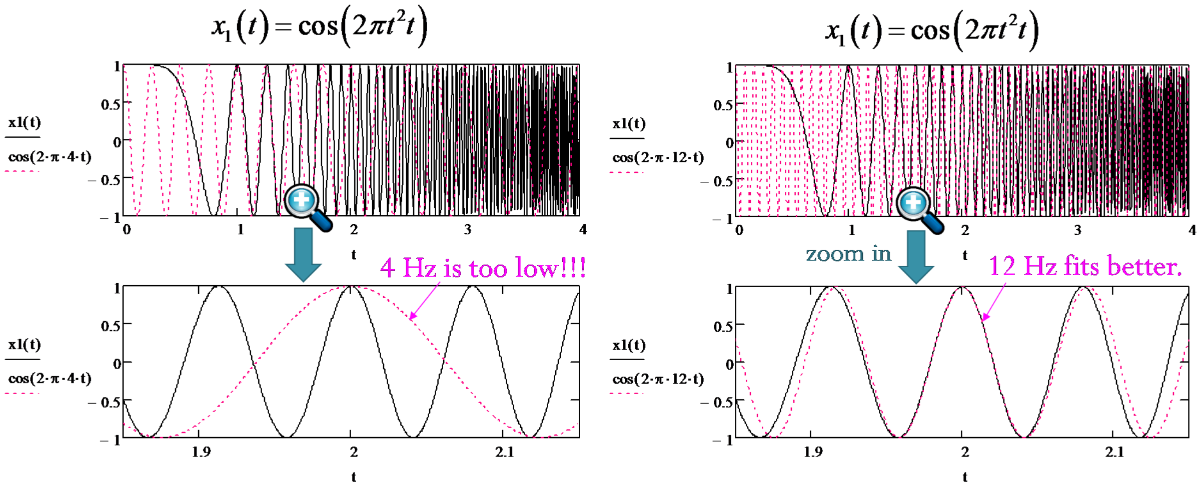

Figure 38: Approximating the frequency of $\cos(2\pi(t^2)t)$ by (a) $\cos(2\pi(4)t)$ and (b) $\cos(2\pi(12)t)$.

(b) Alternatively, around $t = 2$, Figure 38b shows that $\cos(2\pi(12)t)$ seems to provide a good approximation. So, 12 Hz would be a better answer.

Definition 5.13. For generalized sinusoid $A\cos(\theta(t))$, the ***instantaneous frequency***[23] at time t is given by

$A\cos(mt + c)$

$$f(t) = \frac{1}{2\pi}\frac{d}{dt}\theta(t). \tag{75}$$

Around time t, we want to approximate $\theta(t)$ by $mt + c$. From calculus, this can be done by m = slope of $\theta(t)$ at time t $= \frac{d}{dt}\theta(t)$

Example 5.14. For the signal $\cos(2\pi(t^2)t)$ in Example 5.12,

$$\theta(t) = 2\pi(t^2)t$$

and the instantaneous frequency is

$$f(t) = \frac{1}{2\pi}\frac{d}{dt}\theta(t) = \frac{1}{2\pi}\frac{d}{dt}\left(2\pi(t^2)t\right) = 3t^2.$$

In particular, $f(2) = 3 \times 2^2 = 12$.

5.15. The instantaneous frequency formula (75) implies

$f(t) \underset{\frac{1}{2\pi}\frac{d}{dt}}{\overset{2\pi\int}{\rightleftarrows}} \theta(t)$

$$\theta(t) = 2\pi\int_{-\infty}^{t} f(\tau)d\tau = \theta(t_0) + 2\pi\int_{t_0}^{t} f(\tau)d\tau. \tag{76}$$

[23] Although $f(t)$ is measured in hertz, it should not be equated with spectral frequency. Spectral frequency f is the independent variable of the frequency domain, whereas instantaneous frequency $f(t)$ is a time-dependent property of waveforms with exponential modulation.